A Dual-Augmented Block Minimization Framework for Learning with Limited Memory

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Abstract

- In this work, we consider **Empirical Risk Minimization (ERM)** when data size is larger than the memory capacity of machines.
- State-of-the-art batch algorithms become slow due to I/O.
- Online algorithms converge slowly (especially for non-smooth regularizer), while existing distributed approach requires data to fit into memory of several machines.
- We propose a **Block Minimization** framework that generalizes (Yu. et al. 2010) for SVM to that for any convex ERM, which can be integrated with **any convex optimization solver** to achieve global fast convergence in limited-memory condition.

Regularized Empirical Risk Minimization (ERM)

Given a data set $\mathcal{D} = \{(\Phi_n, \boldsymbol{y}_n)\}_{n=1}^N$, the ERM estimates model through

$$\min_{\boldsymbol{w}\in\mathbb{R}^d} F(\boldsymbol{w}) = R(\boldsymbol{w}) + \sum_{n=1}^{N} L_n(\Phi_n \boldsymbol{w}) \quad (1)$$

- $\boldsymbol{w} \in \mathbb{R}^d$ are parameters to be estimated, Φ_n is $p \times d$ feature matrix of n-th sample, and $L_n(.)$, $R(.)$ are loss function and regularizer.

Examples

- **Multiclass Classification:** ($p = |\mathcal{Y}|$, where $\mathcal{Y}$:label set).
  Logistic loss: $L_n(\xi) = \log(\sum_{k\in\mathcal{Y}} \exp(\xi_k)) - \xi_{y_n}$.
  Hinge loss: $L_n(\xi) = \max_{k\in\mathcal{Y}}(1 - \delta_{k,y_n} + \xi_k - \xi_{y_n})$.
- **Multitask Regression:** ($p = K$, where K = #tasks)
  Square loss: $L_n(\xi) = \frac{1}{2}\|\xi - \boldsymbol{y}_n\|^2$.
- **Others:** Ranking, Matrix Completion, Structured Learning, Clustering etc..
- **Regularizers**: L2 norm $\lambda\|\boldsymbol{w}\|^2$, L1 norm $\lambda\|\boldsymbol{w}\|_1$, Group norm $\lambda\|W\|_{\mathcal{G}}$, Nuclear norm $\lambda\|W\|_*$, and etc.

Strong Convexity & Smoothness

- A function $f(x)$ is **strongly convex** iff it is lower bounded by a simple quadratic function

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) + \frac{m}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2 \quad (2)$$

for some constant $m > 0$ and $\forall \boldsymbol{x}, \boldsymbol{y} \in dom(f)$.

- A function $f(x)$ is **smooth** iff it is upper bounded by a simple quadratic function

$$f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) + \frac{M}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2 \quad (3)$$

for some constant $0 \leq M < \infty$ and $\forall \boldsymbol{x}, \boldsymbol{y} \in dom(f)$.

- **Theorem 1:** A convex function $f(.)$ is smooth with parameter M if and only if its convex conjugate $f^*(.)$ is strongly convex with parameter $m = 1/M$.

Dual Form

The dual of ERM problem (1) is of the form

$$\min_{\alpha_n\in\mathbb{R}^p} G(\alpha) = R^*(-\sum_{n=1}^{N}\Phi_n^T\alpha_n) + \sum_{n=1}^{N} L_n^*(\alpha_n). \quad (4)$$

- Block Coordinate Descent on (4) guarantees convergence only for **smooth** $R^*(.)$ (**strongly convex** $R(.)$), which does not hold for most of regularizers.
- Use **Proximal Minimization** to ensure convergence for any convex ERM.

Dual-Augmented Block Minimization

- The Dual-Augmented Lagrangian method (or equivalently, Primal Proximal Minimization) solves a series of **augmented sub-problems**

$$\hat{\boldsymbol{w}}^{t+1} = \arg\min_{\boldsymbol{w}} F(\boldsymbol{w}) + \frac{1}{2\eta_t}\|\boldsymbol{w} - \hat{\boldsymbol{w}}^t\|^2, \quad (5)$$

which, by Theorem 1, has a dual problem of **smooth** $\tilde{R}^*(.)$ since the augmented regularizer $\tilde{R}(\boldsymbol{w})$ is **strongly convex**.

- Let $\mathcal{L}(\boldsymbol{w}, \alpha)$ be the Lagrangian of (5). Our algorithm performs **Block-Coordinate Descent** on dual of (5), which minimizes block of variables α_B via

$$\max_{\alpha_B}\min_{\boldsymbol{w}} \mathcal{L}(\boldsymbol{w}, \alpha) = \min_{\boldsymbol{w}}\max_{\alpha_B} \mathcal{L}(\boldsymbol{w}, \alpha)$$
$$= \min_{\boldsymbol{w}} R(\boldsymbol{w}) + \sum_{n\in B} L_n(\Phi_n\boldsymbol{w}) + \mu_{\bar{B}}^T\boldsymbol{w} + \frac{1}{2\eta_t}\|\boldsymbol{w} - \hat{\boldsymbol{w}}_t\|^2, \quad (6)$$

which requires only data in block B and can be solved via **any solver designed for (1)**, where vector $\mu_{\bar{B}}$ memorizes historical gradient given by data not in B:

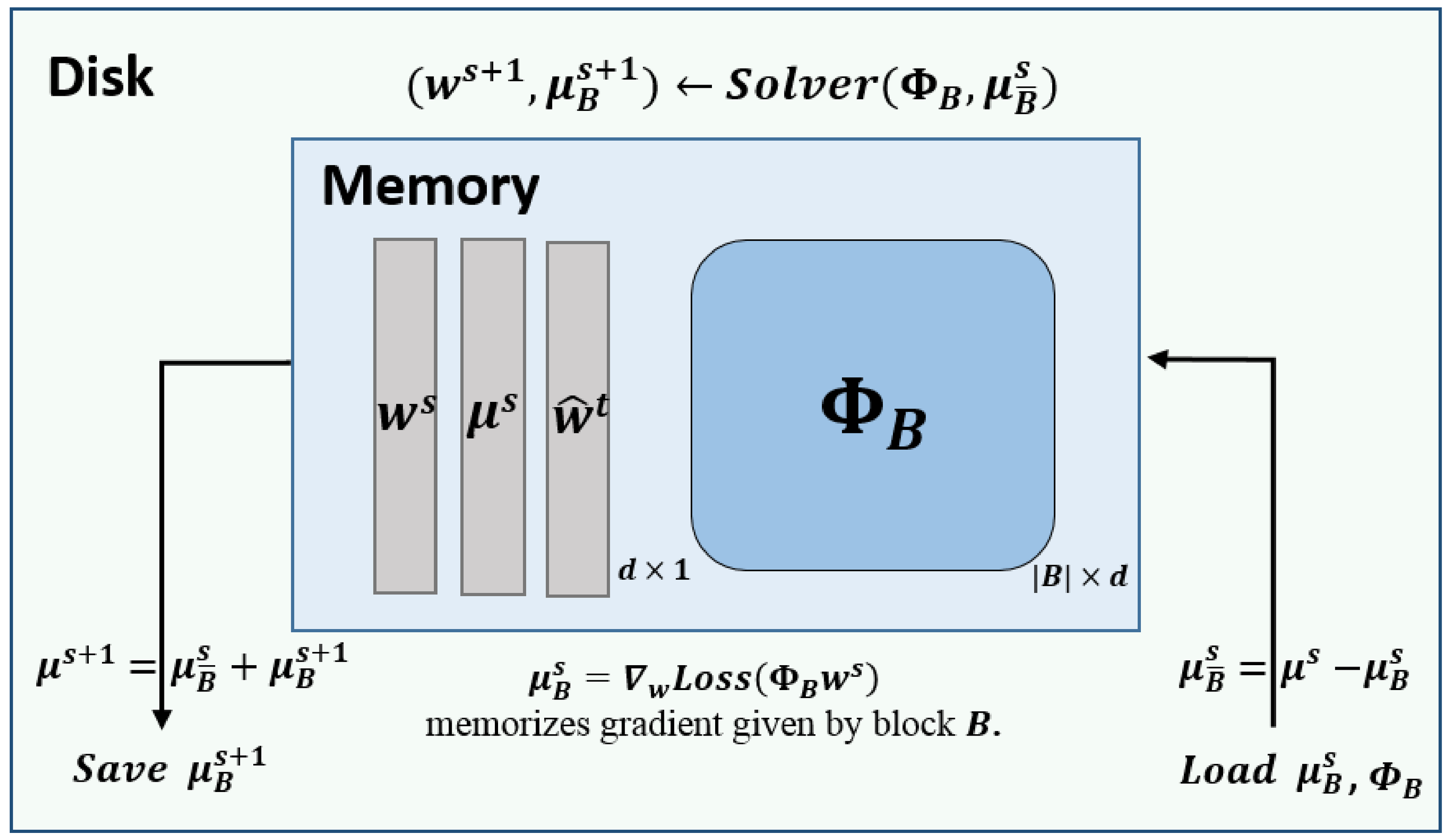
$$\mu_{\bar{B}} = \sum_{n\notin B}\Phi_n^T\alpha_n = \sum_{n=1}^{N}\Phi_n^T\alpha_n - \sum_{n\in B}\Phi_n^T\alpha_n = \mu - \mu_B \quad (7)$$

- After solving block sub-problem (6), we obtain new α_B^* and μ_B^* via

$$\alpha_B^* = \nabla_{\xi_B} Loss(\xi_B^* = \Phi_B\boldsymbol{w}^*) \qquad \mu_B^* = \Phi_B^T\alpha_B^* = \nabla_{\boldsymbol{w}} Loss(\Phi_B\boldsymbol{w}^*). \quad (8)$$

- Only one of α_B or μ_B needs to be maintained. If $d > |B|p$, **maintaining** α_B is cheaper; otherwise, **maintaining** μ_B is more space-efficient.

Algorithmic Framework



Dual-Augmented Block Minimization Algorithm

1. Split data $\mathcal{D}$ into blocks $B_1, B_2, ..., B_K$.
2. Initialize $\hat{\boldsymbol{w}}^0 = \boldsymbol{0}$, $\mu^0 = \boldsymbol{0}$.

for $t = 0, 1, ...$ (outer iteration) **do**
 for $s = 0, 1, ..., S$ **do**
  3.1.1. Draw B uniformly from $B_1, B_2, ..., B_K$.
  3.1.2. Load $\mathcal{D}_B$, μ_B^s (or α_B^s) into memory.
  3.1.3. Solve (6) to obtain $\boldsymbol{w}^*$.
  3.1.4. Maintain μ_B^{s+1} (or α_B^{s+1}) through relation (8).
  3.1.5. Maintain $\mu^{s+1} = \mu_{\bar{B}}^s + \mu_B^{s+1}$.
  3.1.6. Save μ_B^{s+1} (or α_B^{s+1}) out of memory.
 end for
 3.2. $\hat{\boldsymbol{w}}^{t+1} = \boldsymbol{w}^*(\alpha^S)$.
end for

Convergence of Block Minimization

- The dual of (5) takes the form

$$\min_{\alpha_n\in\mathbb{R}^p} \tilde{R}^*(-\sum_{n=1}^{N}\Phi_n^T\alpha_n) + \sum_{n=1}^{N} L_n^*(\alpha_n) \quad (9)$$

where $\tilde{R}^*(.)$ is the convex conjugate of $\tilde{R}(\boldsymbol{w}) = R(\boldsymbol{w}) + \frac{1}{2\eta_t}\|\boldsymbol{w} - \boldsymbol{w}_t\|^2$.

- Since $\tilde{R}(\boldsymbol{w})$ is **strongly convex** with parameter $m = 1/\eta_t$, the convex conjugate $\tilde{R}^*(.)$ is **smooth** with parameter $M = \eta_t$ according to Theorem 1.
- The augmented dual (9) is composite of a **convex, smooth** function plus a **convex, block-separable** function, for which BCD has guaranteed convergence to optimum. In particular, with probability $1 - \rho$

$$\tilde{F}^*(\alpha^s) - \tilde{F}^* \leq \epsilon, \quad \text{for } s \geq \beta K \log(\frac{\tilde{F}^*(\alpha^0) - \tilde{F}^*}{\rho\epsilon}) \quad (10)$$

for some constant $\beta > 0$ if (i) $L_n(.)$ is smooth, or (ii) $L_n(.)$ is polyhedral and $R(.)$ is also polyhedral or smooth. Otherwise, for any convex $L_n(.)$, $R(.)$,

$$\tilde{F}^*(\alpha^s) - \tilde{F}^* \leq \epsilon, \quad \text{for } s \geq \frac{cK}{\epsilon}\log(\frac{\tilde{F}^*(\alpha^0) - \tilde{F}^*}{\rho\epsilon}), \quad (11)$$

with some constant $c > 0$ and probability $1 - \rho$.

Convergence of Overall Procedure

- The sequence $\{\hat{\boldsymbol{w}}^t\}_{t=1}^{\infty}$ produced by Proximal Minimization (5) with $\eta_t = \eta$ and radius of initial level set $\mathcal{R}$ has

$$F(\hat{\boldsymbol{w}}^{t+1}) - F \leq \epsilon, \quad \text{for } t \geq \tau\log(\frac{\omega}{\epsilon}). \quad (12)$$

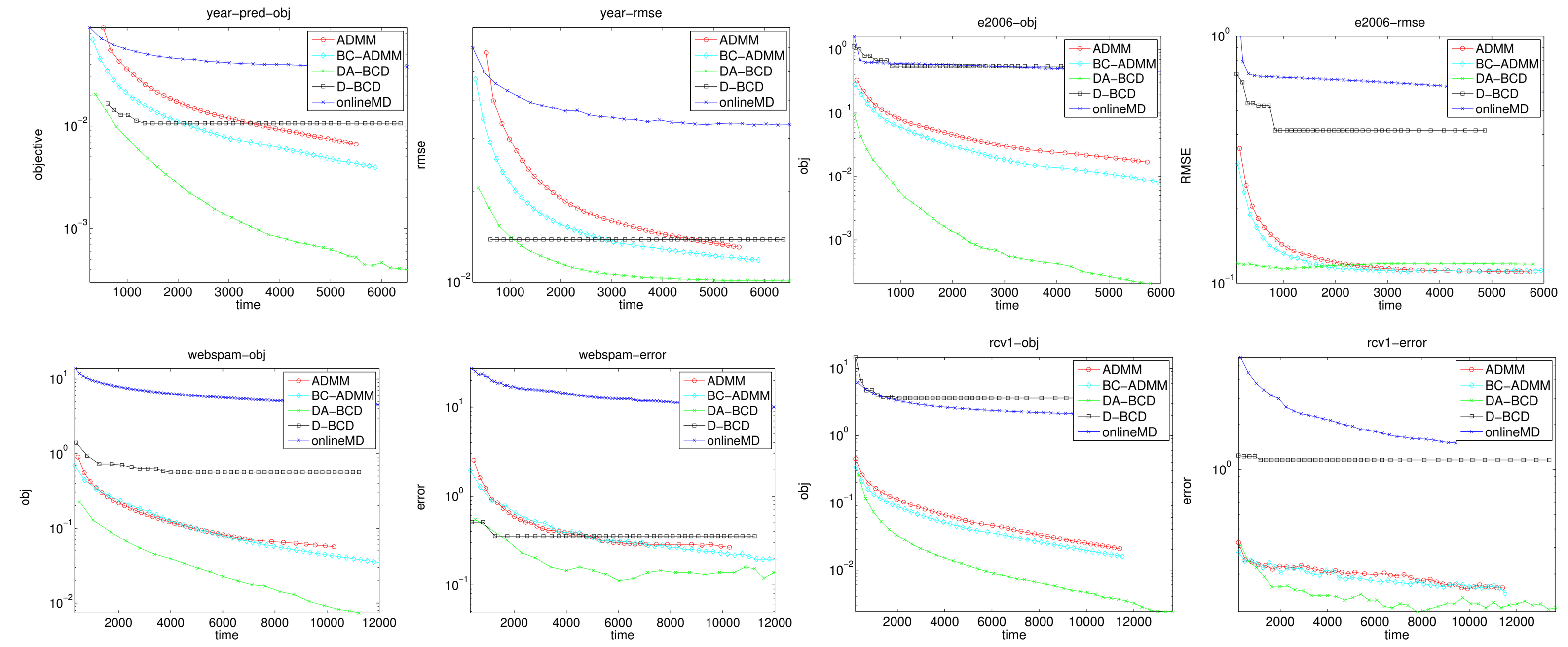
for some constant $\tau, \omega > 0$ if both $L_n(.)$ and $R(.)$ are (i) strictly convex and smooth or (ii) polyhedral. Otherwise, for any convex $F(\boldsymbol{w})$ we have

$$F(\hat{\boldsymbol{w}}^{t+1}) - F \leq \mathcal{R}^2/(2\eta t). \quad (13)$$

- Due to non-expansiveness of proximal operator, we show that solving sub-problem (5) with tolerance ϵ/t suffices for convergence to ϵ overall precision where t is the number of outer iterations required by (12), (13).
- The **overall procedure** requires $O(K\log(1/\epsilon)\log(t/\epsilon))$=$O(K\log^2(1/\epsilon))$ block minimization steps if $L_n(.)$, $R(.)$ are **strictly convex and smooth, or polyhedral**. Otherwise, we need $O(K(1/\epsilon)\log(t/\epsilon)) = O(\frac{K}{\epsilon}\log(1/\epsilon))$ block minimization steps as long as $L_n(.)$ is **smooth**.

Experiments

Data	#train	#test	dimension	#non-zeros	Memory (GB)	Block
webspam	315,000	31,500	680,714	1,174,704,031	20.7	2.07
rcv1	202,420	20,242	7,951,176	656,977,694	12.0	2.20
year-pred	463,715	51,630	2,000	927,893,715	13.7	1.38
E2006	16,087	3,308	30,000	8,088,636	8.08	0.80



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